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Nonlinear $SU(3)$ charged and hypercharged coherent states

Hongyi Fan[1,2,3] **and Guichuan Yu**[3]

[1] CCAST (World Laboratory), PO Box 8730, Beijing 100080, People's Republic of China
[2] Department of Applied Physics, Shanghai Jiao Tong University, Shanghai 200030, People's Republic of China
[3] Department of Material Science and Engineering, University of Science and Technology of China, Hefei, Anhui 230026, People's Republic of China

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Abstract
We introduce the nonlinear $SU(3)$ charged and hypercharged bosonic coherent state in three-mode Fock space. It can be further recast into a compact exponential form. The fermionic case is also discussed.

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1. Introduction

Since the pioneering work of Glauber [1] and Klauder [2], the coherent state (CS) $|\alpha\rangle$ has been widely used in many fields of physics. The photon CS state, defined as the eigenket of the Bose annihilation operator $a|\alpha\rangle = \alpha|\alpha\rangle$, describes quantum mechanically the state of a laser. Many generalized CS, such as the angular moment CS [3], conserved-charged CS [4] and fermionic CS [5], have been established. Another non-Abelian CS, the so-called $SU(3)$ conserved-charged and hypercharged CS [6], can be constructed in three-mode Fock space [6–8], because a three-mode isotropic harmonic oscillator possesses a $SU(3)$ symmetry [9]. In other words, one can represent $SU(3)$ group generators by the three-dimensional harmonic oscillator [10]. The harmonic oscillator representation of the $SU(3)$ generators is [11]

$$S^i = a^\dagger \lambda^i a \qquad (i = 1, 2, \ldots, 8) \tag{1}$$

with $a^\dagger$ and a defined as

$$a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \qquad a^\dagger = (\, a_1^\dagger \quad a_2^\dagger \quad a_3^\dagger \,)\,.$$

λ^i is the Gell-Mann matrix satisfying

$$[\lambda^i, \lambda^j] = 2\mathrm{i} f^{ijk} \lambda^k \tag{2}$$

where f^{ijk} are the group structure constants. The explicit forms of S^i are

$$\begin{aligned} S^1 &= a_1^\dagger a_2 + a_2^\dagger a_1, & S^2 &= -\mathrm{i}a_1^\dagger a_2 + \mathrm{i}a_2^\dagger a_1 \\ S^3 &= a_1^\dagger a_1 - a_2^\dagger a_2 & S^4 &= a_1^\dagger a_3 + a_3^\dagger a_1 \\ S^5 &= -\mathrm{i}a_1^\dagger a_3 + a_3^\dagger a_1 & S^6 &= a_2^\dagger a_3 + a_3^\dagger a_2 \\ S^7 &= -\mathrm{i}a_2^\dagger a_3 + a_3^\dagger a_2 & S^8 &= \frac{1}{\sqrt{3}}(a_1^\dagger a_1 + a_2^\dagger a_2 - 2a_3^\dagger a_3) \end{aligned}$$

with $[a_i, a_j^\dagger] = \delta_{ij}$. The representations of charge operator Q and hypercharge operator Y are given by

$$Q = \tfrac{1}{3}(2a_1^\dagger a_1 - a_2^\dagger a_2 - a_3^\dagger a_3) \tag{3}$$

$$Y = \frac{1}{\sqrt{3}} S^8. \tag{4}$$

Due to the fact that Q and Y commute with the product of three modes of annihilation operators

$$[Q, a_1a_2a_3] = 0 \qquad [Y, a_1a_2a_3] = 0 \qquad [Q, Y] = 0 \tag{5}$$

there must exist their common eigenvector, denoted by $|z, y, q\rangle$, which satisfies

$$Q|z, y, q\rangle = q|z, y, q\rangle \tag{6}$$

$$Y|z, y, q\rangle = y|z, y, q\rangle \tag{7}$$

$$a_1a_2a_3|z, y, q\rangle = z|z, y, q\rangle. \tag{8}$$

In [6] the explicit form of $|z, y, q\rangle$ is obtained:

$$|z, y, q\rangle = N_{qy} \sum_{l=0}^{\infty} \frac{z^l}{[l!(l+y+q)!(l+2y-q)!]^{1/2}} |l+y+q, l+2y-q, l\rangle \tag{9}$$

where $|l+y+q, l+2y-q, l\rangle$ is a three-mode Fock state, and N_{qy} is the normalization factor:

$$N_{qy} = \left[\sum_{l=0}^{\infty} \frac{(|z|^2)^l}{l!(l+y+q)!(l+2y-q)!}\right]^{-1/2}. \tag{10}$$

Another approach to obtaining $|z, y, q\rangle$ (up to a constant factor) is by starting from the usual three-mode CS $|\alpha, \beta, \gamma\rangle$:

$$|\alpha, \beta, \gamma\rangle = \exp(\alpha a_1^\dagger + \beta a_2^\dagger + \gamma a_3^\dagger)|000\rangle \tag{11}$$

then letting

$$\begin{aligned} &\alpha = \lambda_1 \exp[\mathrm{i}(\theta + 2\phi + \varphi)] \qquad \beta = \lambda_2 \exp[\mathrm{i}(\theta - \phi + \varphi)] \\ &\gamma = \lambda_3 \exp[\mathrm{i}(\theta - \phi - 2\varphi)] \end{aligned} \tag{12}$$

and by considering a suitable average over two $U(1)$ phases according to the following expression:

$$\begin{aligned} \frac{1}{4\pi^2} &\int_0^{2\pi} \mathrm{d}\phi \int_0^{2\pi} \mathrm{d}\varphi \exp(-\mathrm{i}3q\phi) \exp(-\mathrm{i}3y\varphi)|\alpha, \beta, \gamma\rangle \\ &= \mathrm{e}^{3\mathrm{i}y\theta} \lambda_1^{y+q} \lambda_2^{2y-q} \sum_l \frac{z^l}{[l!(l+y+q)!(l+2y-q)!]^{\frac{1}{2}}} \\ &\times |l+y+q, l+2y-q, l\rangle = |z, y, q\rangle \end{aligned} \tag{13}$$

where we have demanded $3q$ and $3y$ being integers, and

$$z = \lambda_1\lambda_2\lambda_3 \exp(-\mathrm{i}\theta). \tag{14}$$

Once we introduce 'charge' and 'hypercharge' by defining three types of quanta (a_1, a_2, a_3) possessing fractional 'charge' $\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}$ and 'hypercharge' $\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}$, which corresponds to the $SU(3)$ quark model in elementary particle physics [11], we can call $|z, y, q\rangle$ the $SU(3)$ charged and hypercharged CS. For example, if a particle carries charge $q = 1$, supercharge $y = 1$, then from (9) we have the corresponding $SU(3)$ charged and hypercharged CS:

$$|z, 1, 1\rangle = N_{11} \sum_{l=0}^{\infty} \frac{z^l}{[l!(l+1)!(l+2)!]^{\frac{1}{2}}} |l+2, l+1, l\rangle$$

which states that the quark content of this particle is uud (two u quarks and one d quark). By considering that quarks are actually fermions, the $SU(3)$ charged and hypercharged fermionic CS is also constructed in [6].

Recently, another route to generalizing the concept of CS leads to the so-called nonlinear coherent state (NLCS) [12–17]. One special NLCS could be generated as the stationary state of the centre-of-mass motion of a laser-driven trapped ion far from the Lamb–Dicke limit [13]. The NLCS is defined as

$$f(N)a|f, \alpha\rangle = \alpha|f, \alpha\rangle \tag{15}$$

where $f(N)$ is an operator-valued function of the number operator $N = a^\dagger a$. Another kind of NLCS is defined to satisfy the following eigenvalue equation:

$$f(N)a^n|\alpha, f\rangle = \alpha|\alpha, f\rangle. \tag{16}$$

For example, one can prove that the single-mode squeezed vacuum state and squeezed one-photon state satisfy equation (16):

$$\frac{1}{N+1} a^2 S(z)|0\rangle = \alpha S(z)|0\rangle \tag{17}$$

$$\frac{1}{N+1} a^2 S(z)|1\rangle = \alpha S(z)|1\rangle \tag{18}$$

where $S(z)$ is the one-mode squeezing operator:

$$S(z) = \exp\left(\frac{z}{2}a^2 - \frac{z^*}{2}a^{\dagger 2}\right) \qquad z = r\mathrm{e}^{\mathrm{i}\theta} \qquad \alpha = \mathrm{e}^{\mathrm{i}\theta}\tanh r. \tag{19}$$

A question thus naturally arises: how to construct a nonlinear $SU(3)$ charged and hypercharged coherent state (NLSUCHCS)? In section 2 we construct bosonic NLSUCHCS, which can be further put in a compact exponential form. In section 3 we extend the bosonic NLSUCHCS to the fermionic case, because the quark is a fermion.

2. $SU(3)$ nonlinear charged and hypercharged coherent state—bosonic case

By analogy with the single-mode NLCS, we define the NLSUCHCS $|z, y, q\rangle_f$ as the common eigenvector of the following three operators:

$$f(N_1, N_2, N_3)a_1a_2a_3|z, y, q\rangle_f = z|z, y, q\rangle_f \qquad N_i = a_i^\dagger a_i \tag{20}$$

$$Q|z, y, q\rangle_f = q|z, y, q\rangle_f \tag{21}$$

$$Y|z, y, q\rangle_f = y|z, y, q\rangle_f \tag{22}$$

since

$$[f(N_1, N_2, N_3)a_1a_2a_3, Q] = 0 \qquad [f(N_1, N_2, N_3)a_1a_2a_3, Y] = 0.$$

Here the annihilator of three modes multiplied by $f(N_1, N_2, N_3)$ means that the annihilating process is field intensity dependent. Expanding $|z, y, q\rangle_f$ in the Fock basis

$$|z, y, q\rangle_f = \sum_{m,n,l=0}^{\infty} C_{mnl}|mnl\rangle \tag{23}$$

where $|mnl\rangle$ is the three-mode number state:

$$|mnl\rangle = \frac{a_1^{\dagger m} a_2^{\dagger n} a_3^{\dagger l}}{\sqrt{m!n!l!}}|000\rangle$$

and then acting Q and Y on $|z, y, q\rangle_f$, as a result of (21) and (22) we see that

$$n - l = 2y - q \qquad m - l = y + q. \tag{24}$$

Next, operating $f(N_1, N_2, N_3)a_1a_2a_3$ on equation (23), we get the recursive relation

$$\begin{aligned} C_{m,n,l} &= \frac{z}{f(m-1, n-1, l-1)\sqrt{mnl}} C_{m-1,n-1,l-1} \\ &= z^l \sqrt{\frac{(m-l)!(n-l)!}{m!n!l!}} \prod_{k=1}^{l} \frac{1}{f(m-k, n-k, l-k)} C_{m-l,n-l,0} \end{aligned} \tag{25}$$

where, without loss of generality, we have assumed $m \geqslant l, n \geqslant l$. Thus $|z, y, q\rangle_f$ is expressed as

$$\begin{aligned} |z, y, q\rangle_f = C_0 \sum_{l=0}^{\infty} & \frac{(za_1^\dagger a_2^\dagger a_3^\dagger)^l (m-l)!(n-l)!}{m!n!l!} \\ & \times \prod_{k=1}^{l} \frac{1}{f(m-k, n-k, l-k)} |m-l, n-l, 0\rangle \end{aligned} \tag{26}$$

where $C_0 \equiv C_{m-l,n-l,0}$ is the normalization constant. It is desirable to put NLSUCHCS into a neat exponential form, so from (26) we have

$$\begin{aligned} |z, y, q\rangle_f = C_0 \sum_{l=0}^{\infty} & \frac{(za_1^\dagger a_2^\dagger a_3^\dagger)^l}{[(m-l+1)\cdots m][(n-l+1)\cdots n]l!} \\ & \times \prod_{k=1}^{l} \frac{1}{f(m-k, n-k, l-k)} |m-l, n-l, 0\rangle \end{aligned} \tag{27}$$

where

$$\begin{aligned} & \frac{1}{[(m-l+1)\cdots m][(n-l+1)\cdots n]} \\ & \qquad \times \prod_{k'=1}^{l} \frac{1}{f(N_1+l-k', N_2+l-k', N_3+l-k')} |m-l, n-l, 0\rangle \\ & \quad = \prod_{k=1}^{l} \frac{1}{(m-l+k)(n-l+k) f(N_1-1+k, N_2-1+k, N_3-1+k)} \\ & \qquad \times |m-l, n-l, 0\rangle \\ & \quad = \prod_{k=1}^{l} \frac{1}{(N_1+k)(N_2+k) f(N_1-1+k, N_2-1+k, N_3-1+k)} \\ & \qquad \times |m-l, n-l, 0\rangle. \end{aligned} \tag{28}$$

Substituting (28) into (27) and using the operator identity

$$(a_1^\dagger a_2^\dagger a_3^\dagger)^l \prod_{k=1}^{l} f(N_1+k, N_2+k, N_3+k) = [f(N_1, N_2, N_3)a_1^\dagger a_2^\dagger a_3^\dagger]^l \tag{29}$$

we see that the state (26) can be written as the following compact exponential form:

$$|z, y, q\rangle_f = C_0 \sum_{l=0}^{\infty} \frac{z^l}{l!} \left[\frac{1}{N_1 N_2 f(N_1-1, N_2-1, N_3-1)} a_1^\dagger a_2^\dagger a_3^\dagger \right]^l |y+q, 2y-q, 0\rangle. \tag{30}$$

One can confirm this result by noticing

$$af(N) = f(N+1)a \qquad a^\dagger f(N) = f(N-1)a^\dagger \tag{31}$$

$$\left[f(N_1, N_2, N_3)a_1 a_2 a_3, \ \frac{1}{N_1 N_2 f(N_1-1, N_2-1, N_3-1)} a_1^\dagger a_2^\dagger a_3^\dagger \right] = 1 \tag{32}$$

so equation (20) is checked. The exponential form (30) is also convenient for us to check (21). Due to

$$[Q, a_1^\dagger a_2^\dagger a_3^\dagger] = 0 \qquad [Y, a_1^\dagger a_2^\dagger a_3^\dagger] = 0$$

so

$$Q|z, y, q\rangle_f = C_0 \exp\left[\frac{z}{N_1 N_2 f(N_1-1, N_2-1, N_3-1)} a_1^\dagger a_2^\dagger a_3^\dagger \right] \times Q|y+q, 2y-q, 0\rangle = q|z, y, q\rangle_f.$$

In particular, when $f(N_1, N_2, N_3)$ is set to 1, then $|z, y, q\rangle_f$ reduces to the ordinary $|z, y, q\rangle$:

$$|z, y, q\rangle = C_0 \exp\left[\frac{z}{N_1 N_2} a_1^\dagger a_2^\dagger a_3^\dagger \right] |y+q, 2y-q, 0\rangle. \tag{33}$$

In fact, by operating a_3 on $|z, y, q\rangle$ we have

$$a_3|z, y, q\rangle = C_0 \left[a_3, \exp\left[\frac{z}{N_1 N_2} a_1^\dagger a_2^\dagger a_3^\dagger \right] \right] |y+q, 2y-q, 0\rangle = \frac{z}{N_1 N_2} a_1^\dagger a_2^\dagger |z, y, q\rangle.$$

Then using $a_1 a_2 \frac{1}{N_1 N_2} a_1^\dagger a_2^\dagger = 1$, we see $a_1 a_2 a_3 |z, y, q\rangle = z|z, y, q\rangle$. For another case, when

$$f(N_1, N_2, N_3) = \frac{1}{\sqrt{(N_1+1)(N_2+1)(N_3+1)}}$$

then $|z, y, q\rangle_f$ becomes

$$|z, y, q\rangle_f = C_0 \exp\left[\frac{z\sqrt{N_3}}{\sqrt{N_1 N_2}} a_1^\dagger a_2^\dagger a_3^\dagger \right] |y+q, 2y-q, 0\rangle. \tag{34}$$

Recall that the Susskind–Glogower phase operator is defined as $\frac{1}{\sqrt{(N_i+1)}} a_i$, $i = 1, 2, 3$, so equation (34) is the eigenvector of the three-mode Susskind–Glogower phase operator.

The NLCS can be generalized to the m-mode case. Its exponential form is

$$|z, \{G\}\rangle = \exp\left[\frac{z}{N_1 \ldots N_{m-1} f(N_1-1, \ldots, N_m-1)} a_1^\dagger \ldots a_m^\dagger \right] |\ldots, 0\rangle \tag{35}$$

where $\{G\}$ denotes the set of other good quantum numbers, which are eigenvalues of some independent operators commuting with $f(N_1, N_2, \ldots, N_m)a_1 a_2 \ldots a_m$, $|\ldots, 0\rangle$, which is an m-mode Fock state with the mth mode being vacant. $|z, \{G\}\rangle$ obeys the eigenvector equation:

$$f(N_1, N_2, \ldots, N_m)a_1 a_2 \ldots a_m |z, \{G\}\rangle = z|z, \{G\}\rangle. \tag{36}$$

3. $SU(3)$ nonlinear charged, hypercharged coherent state—fermionic case

The above discussion can be extended to the fermionic case. Introducing the following generators:

$$G^i = b^\dagger \lambda^i b \tag{37}$$

where $b^\dagger$ and b denote

$$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \qquad b^\dagger = (\, b_1^\dagger \quad b_2^\dagger \quad b_3^\dagger \,)$$

and b_α are Fermi operators satisfying the anticommutative relation

$$\{b_\alpha, b_\beta^\dagger\} = \delta_{\alpha\beta} \qquad \alpha, \beta = 1, 2, 3. \tag{38}$$

With use of the operator identity

$$[A, BC] = \{A, B\}C - B\{A, C\} \tag{39}$$

it is easy to verify

$$[G^i, G^j] = 2\mathrm{i} f^{ijk} G^k. \tag{40}$$

Similar to equations (3) and (4), we introduce the fermionic NLSUCHCS:

$$Q' = \frac{1}{2}G^3 + \frac{1}{2\sqrt{3}}G^8 = \frac{1}{3}(2b_1^\dagger b_1 - b_2^\dagger b_2 - b_3^\dagger b_3) \tag{41}$$

$$Y' = \frac{1}{\sqrt{3}}G^8 = \frac{1}{3}(b_1^\dagger b_1 + b_2^\dagger b_2 - 2b_3^\dagger b_3) \tag{42}$$

which satisfy

$$[Q', b_1 b_2 b_3] = 0 \qquad [Y', b_1 b_2 b_3] = 0 \qquad [Q', Y'] = 0. \tag{43}$$

Let $N_{\alpha b} = b_\alpha^\dagger b_\alpha$. By analogy with (20)–(22), we can construct the common eigenvector of Q', Y', and $f(N_{1b}, N_{2b}, N_{3b}) b_1 b_2 b_3$, which is denoted by

$$Q'|\xi, y, q\rangle_f = q|\xi, y, q\rangle_f \qquad Y'|\xi, y, q\rangle_f = y|\xi, y, q\rangle_f \tag{44}$$

$$f(N_{1b}, N_{2b}, N_{3b}) b_1 b_2 b_3 |\xi, y, q\rangle_f = \xi|\xi, y, q\rangle_f. \tag{45}$$

Since

$$[f(N_{1b}, N_{2b}, N_{3b}) b_1 b_2 b_3]^2 |\xi, y, q\rangle_f = -\xi^2 |\xi, y, q\rangle_f = 0$$

ξ must be a Grassmann number with property $\xi^2 = 0$. The Grassmann number anticommutes with a single Fermi operator. We can derive the explicit form of $|\xi, y, q\rangle_f$ by expanding it in Fock basis:

$$|\xi, y, q\rangle_f = \sum_{n,m,l} |m, n, l\rangle C_{mnl} = \sum_{n,m,l} b_1^m b_2^n b_3^l |0, 0, 0\rangle C_{mnl} \qquad m, n, l = 0, 1.$$

Substituting it into the eigenvector equation (45) we find that

$$\xi C_{000} = -f(0, 0, 0) C_{111}$$
$$\xi C_{mnl} = 0 \qquad m \cdot n \cdot l \neq 0.$$

Then, by taking into account (44) we obtain

$$qC_{100} = \tfrac{2}{3}C_{100} \qquad qC_{011} = -\tfrac{2}{3}C_{011} \qquad qC_{110} = \tfrac{1}{3}C_{110} \qquad qC_{101} = \tfrac{1}{3}C_{101}$$
$$qC_{010} = -\tfrac{1}{3}C_{010} \qquad qC_{001} = -\tfrac{1}{3}C_{001} \qquad qC_{000} = 0 \qquad qC_{111} = 0.$$
$$yC_{110} = \tfrac{2}{3}C_{110} \qquad yC_{001} = -\tfrac{2}{3}C_{001} \qquad yC_{010} = \tfrac{1}{3}C_{010} \qquad yC_{100} = \tfrac{1}{3}C_{100}$$
$$yC_{011} = -\tfrac{1}{3}C_{011} \qquad yC_{101} = -\tfrac{1}{3}C_{101} \qquad yC_{000} = 0 \qquad yC_{111} = 0.$$

Hence $|\xi, y, q\rangle_f$ is

$$\begin{array}{ll} |\xi, \frac{1}{3}, \frac{2}{3}\rangle = \xi|1,0,0\rangle & |\xi, -\frac{1}{3}, -\frac{2}{3}\rangle = \xi|0,1,1\rangle \\ |\xi, \frac{2}{3}, \frac{1}{3}\rangle = \xi|1,1,0\rangle & |\xi, -\frac{2}{3}, -\frac{1}{3}\rangle = \xi|0,0,1\rangle \\ |\xi, -\frac{1}{3}, \frac{1}{3}\rangle = \xi|1,0,1\rangle & |\xi, \frac{1}{3}, -\frac{1}{3}\rangle = \xi|0,1,0\rangle \\ |\xi, 0, 0\rangle = \xi|0,0,0\rangle & |\xi, 0, 0\rangle = f(0,0,0)|0,0,0\rangle + \xi|1,1,1\rangle \end{array} \tag{46}$$

or written in an exponential form which is similar in form to (30)

$$|\xi, y, q\rangle_f = \exp\left[\frac{\xi}{N_{1b}N_{2b}f(1-N_{1b}, 1-N_{2b}, 1-N_{3b})}b_1^\dagger b_2^\dagger b_3^\dagger\right]|y+q, 2y-q, 0\rangle. \tag{47}$$

This can be checked by using

$$bN_b = (1-N_b)b \qquad b^\dagger N_b = (1-N_b)b^\dagger \qquad Nb^\dagger = b^\dagger(1-N_b) \tag{48}$$

and

$$\left[f(N_{1b}, N_{2b}, N_{3b})b_1b_2b_3, \frac{\xi}{N_{1b}N_{2b}f(1-N_{1b}, 1-N_{2b}, 1-N_{3b})}b_1^\dagger b_2^\dagger b_3^\dagger\right] = \xi. \tag{49}$$

Recall that in [18] the CS formalism is set up for presenting quark and gluon cross sections. We wish some experimental consequence can be calculated on the basis of such nonlinear $SU(3)$ charged and hypercharged coherent states.

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